Viscosity solutions theory for Hamilton-Jacobi equations and traffic flow

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Abstract

This report has been written by Jérémy Firozaly, supervised by Cyril Imbert and Régis Monneau, who are respectively CNRS research director at LAMA laboratory in Créteil University and chief engineer at CERMICS laboratory in Ponts ParisTech school. It is both part of final year project at ENSTA ParisTech (PFE internship) and of master project at UPMC University in the framework of the "Numerical Analysis and Partial Derivative Equations" M2. It aims at introducing the students to the world of research by testing their capacity to work independently and in my case, at giving me the basical knowledge and culture in the field of my future PhD thesis that will also be supervised by Cyril Imbert and Régis Monneau.

Cyril Imbert and Régis Monneau are particularly interested in viscosity solutions for Hamilton-Jacobi equations in the framework of traffic flow mathematical modeling. Namely, they consider different models of pursuit laws like 1D straight line traffic or divergent junctions. In my PhD thesis, we will especially consider 1D models and we will try to show that such models, generally taking the form of infinite system of non-linear coupled ODE's, are well posed in the viscosity sense. In a second time, we will try to transpose such models to a bigger scale, typically by an homogenization process, to study the effect of lights and drivers' reaction time on global traffic flows. There is currently no mathematical result in the classical literature on this precise subject, apart from their own researches.

During my project, I first studied some features of viscosity solutions theory for Hamilton-Jacobi equations and then tried to adapt it to a non-local equation which will be useful for the models that will be considered in my PhD thesis. I also focused on the modeling process for the pursuit laws that I might consider during my PhD thesis. The present report aims at giving a structured overview of the knowledges I have acquired about viscosity solutions theory and the modeling of pursuit laws.

key-words:

-viscosity solutions theory;

-Hamilton-Jacobi equations;

-microscopic and macroscopic scales;

-homogenization.

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1 Introduction

1.1 Some interesting features of viscosity solutions theory

The ideas presented here are inspired from [5]. Those aspects are clearly not exhaustive and many examples or applications of this theory can be found in the literature, see [7] for example.

1.1.1 Definition and selection of solutions

Let us consider, the following equation with its corresponding Dirichlet boundary conditions, in one dimension, on $I = [0;1]$:

$$\begin{cases} |u'(x)| = 1 & \forall x \in I \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

Mathematicians are usually looking for pleasant properties when they analyse PDEs: existence and uniqueness of solutions, and if applicable, smoothness with respect to some parameters or boundary conditions.

In such a simple example, we directly encounter many problems for these properties. By invoking Rolle's Theorem, we can see that if there exists a smooth solution u, there will exist $c \in]0;1[$ such that $u'(c) = 0 \neq 1$ which is contradictory.

The highly non-linear character of the equation, precisely the absolute value on the derivative, prevents from using the theory of distributions. If we look for Lipschitz solutions on I, that is $u \in W^{1,\infty}(I)$, we can define solutions of (1) in the sense "almost everywhere" by Rademacher's Theorem. However, as shown in the picture below, all solutions verifying the boundary conditions with slopes varying between the two values $+1$ and -1 are solutions. We here lose the uniqueness of the solution.

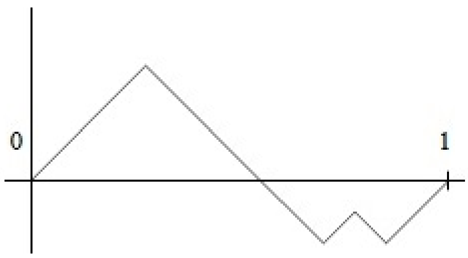

Figure 1: Example of Lipschitz solution.

Using the "vanishing viscosity" method, that is, adding a term of the form $-\varepsilon u''(x)$ in the left hand side of (1), without changing the boundary conditions, we can obtain a unique smooth solution, u_ε say, for each $\varepsilon > 0$. A natural question is: does u_ε converge for vanishing ε, in which sense, and is the possible limit a solution of (1) in a sense that could ensure uniqueness?

The viscosity solutions theory will allow us to pass to the limit in a precise sense, for a huge class of equations containing the previous one: Hamilton-Jacobi equations. It will then provide a correct framework to obtain existence and uniqueness for solutions and define the "derivatives" for weak and non-smooth solutions. Let us consider, for information purposes only (it will not be studied in this report), another interest of viscosity solutions theory.

1.1.2 Boundary conditions

Let us consider another equation, linear this time, that will fit into the framework of Hamilton-Jacobi equations. Let Ω be the unit open disc of $\mathbb{R}^2$, and (e_1, e_2) its canonical basis, we look at a transport equation with Dirichlet boundary conditions (for a given function $g \in C(\partial\Omega)$):

$$\begin{cases} -e_1.\nabla u(x) = 0 & \forall x \in \Omega \\ u(x) = g(x) & \forall x \in \partial\Omega. \end{cases} \tag{2}$$

The solution u is supposed to be constant along the vector e_1 and therefore, we cannot really impose a Dirichlet condition on the whole boundary of Ω.

In fact, we can only impose such a condition on $\partial\Omega \cap \{(x, y) \in \mathbb{R}^2, x \leq 0\}$ (or $\partial\Omega \cap \{(x, y) \in \mathbb{R}^2, x \geq 0\}$). Again using vanishing viscosity method, we can similarly find a unique smooth solution u_ε for an approximate problem at fixed ε (with a Dirichlet boundary condition on the whole frontier). A good solution for (2) would be the possible limit of the sequence, for vanishing ε. The huge interest of using viscosity solutions theory is that the passage to the limit is not clear at all, as the first problem is not very well defined.

Therefore, viscosity solutions theory seems to be useful for lots of reasons, such as, giving a sense to solutions or boundary conditions for some equations. As made precise earlier, this theory has a lot of other interests, that I may discover during my future PhD thesis. Some aspects presented in the introduction will be further discussed in part 2.

1.2 An overview on traffic flow scales

There are typically two scales to describe road traffic on a straight line: the microscopic scale where we follow the time evolution of each driver and the macroscopic one where we rather focus on time evolution of densities in global traffic, that is on flux circulations. Here are pictures showing those two scales (the second picture is just here to illusrate what is the macroscopic scale generically as it actually describes a more general situation than the framework of simple straight line):

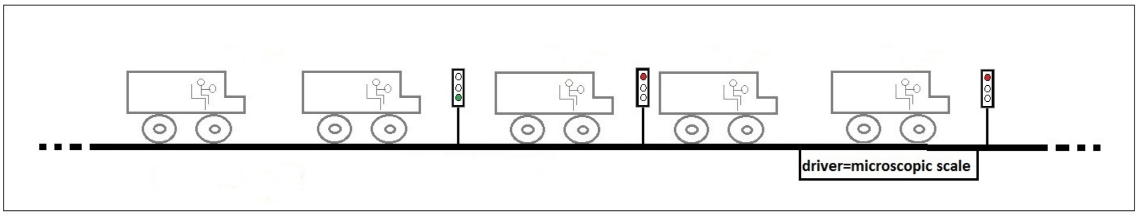

Figure 2: View of the microsopic scale considered in our study.

Figure 3: Schematic view of a generic macrosopic scale from sytadin.fr.

When possible, the passage from the microscopic to the macroscopic scale is called homogenization.

Those features will be more precisely described in part 4.

Plan. In the first part of this report, we will define what viscosity solutions for Hamilton-Jacobi equations are, especially by giving the useful assumptions on the Hamiltonians and domains. We will go through notions from comparison principle which leads to uniqueness, to Perron's method which leads to existence of viscosity solutions. We will also deal with the stability of these equations, that is, passages to the limit for viscosity solutions. Some results will be common for first order and second order Hamilton-Jacobi equations and others will be specific to the first order, which is basically the one concerning traffic flow.

In the second part, we will study a particular model related to traffic flow, which can be seen as a non-local version of Hamilton-Jacobi equation. We will adapt some theorems evoked in the first part, namely comparison principles.

In the last part, we will give some aspects on traffic flow modeling and will introduce some features of my future PhD thesis such as homogenization. We will first introduce assumptions on microscopic traffic flow model in order to modelize lights, drivers' reaction times and corresponding velocities. Secondly, we will briefly present the way to go from this microscopic model to a macroscopic point of view, especially by embedding it into a single PDE.

2 Introduction to viscosity solutions theory for Hamilton-Jacobi equations

This section is largely inspired by the courses I have received from Hasnaa Zidani and Olivier Bokanowski at ENSTA ParisTech about wavefront propagation (see [1]), by Cyril Imbert and Jérôme Droniou's lecture notes on variational and viscosity solutions for non-linear PDEs (see [2]) and by the famous "User's guide to viscosity solutions" from M. Crandall, H. Ishii, and P-L. Lions (see [3]). We will introduce key concepts about this theory with a particular point of view, based on the meetings we had with Cyril Imbert and Régis Monneau.

To make the notations clear, subscripts x and t indicate respectively the partial derivative with respect to the spatial coordinate, $x \in \mathbb{R}$, at fixed time $t \in \mathbb{R}_+$ and the partial derivative with respect to the time variable, $t \in \mathbb{R}_+$, at fixed $x \in \mathbb{R}$. Ω will denote an open bounded subset of $\mathbb{R}^n$ equipped with its euclidian norm $||.||$. $S(n)$ will denote the set of symmetric $n \times n$ matrices with its usual partial order. The scalar PDEs that we will study are referred to as "Hamilton-Jacobi equations". Their generic solution will be denoted by u, its gradient by Du and its Hessian matrix by D^2u. For non smooth u, viscosity solutions theory will allow us to give a meaning for gradient and Hessian matrix. We will then be able to study ranges of equations that do not have classical smooth solutions.

2.1 Definitions, general concepts

Hamilton-Jacobi equations are of the form:

$$F(x, u(x), Du(x), D^2u(x)) = 0 \qquad \forall x \in \Omega \tag{3}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ satisfies the following assumptions:

$$\begin{cases} F \text{ is continuous,} \\ \forall (x, r, s, p, X) \in \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times S(n), \qquad r \leq s \Rightarrow F(x, r, p, X) \leq F(x, s, p, X) \\ \forall (x, r, p, X, Y) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \times S(n), \quad Y \leq X \Rightarrow F(x, r, p, X) \leq F(x, s, p, Y). \end{cases} \tag{4}$$

Such an Hamiltonian F is said to be proper. The third assumption is called "degenerate ellipticity". Unless otherwise stated, F will be supposed proper in the whole section.

Let us suppose that $u \in C^2(\Omega, \mathbb{R})$, meaning that u is a classical solution of (3), and consider $\varphi \in C^2(\Omega, \mathbb{R})$ such that $u - \varphi$ has a local minimum at $x_0 \in \Omega$ say.

This implies that: $Du(x_0) = D\varphi(x_0)$ and $D^2u(x_0) \geq D^2\varphi(x_0)$. By degenerate ellipticity of F, we end to: $F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$ which does not involve any derivative of u. That comment suggests the following definition:

Definition 2.1 (Viscosity solutions for Hamilton-Jacobi equations)**.** *Assume* (4) *and let $u : \Omega \to \mathbb{R}$ be a locally bounded function.*

- *The function u is said to be a viscosity supersolution (resp. a subsolution) of* (3) *if u is lower semi-continuous (resp. upper semi-continuous) and for all $x_0 \in \Omega$ and all test function $\varphi \in C^2(\Omega, \mathbb{R})$ such that $u - \varphi$ attains a local minimum (resp. a local maximum) at x_0, then we have:*

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \quad (\textit{resp. } \leq).$$

- *A function u is said to be a viscosity solution of* (3) *if it is both a subsolution and a supersolution.*

A viscosity solution is hence continuous by definition. By construction, a classical solution will be a viscosity solution.

Remark 2.2 (About the local extremum). *The assumptions about local minimum (resp. maximum) can be replaced by strict local minimum (resp. maximum) by substracting (resp. adding) a smooth term* $||x - x_0||^4$ *on test functions. This term does not change their derivatives of first and second order at* x_0.

There is an immediate corollary:

Corollary 2.3 (Classical solutions VS viscosity solutions).
Let $u \in C^2(\Omega, \mathbb{R})$. *Then,* u *is a classical solution of* (3) *if and only if* u *is a viscosity solution of* (3).

Proof. The direct sense is clear by construction. Let us focus on the converse sense. As u is smooth, Du and D^2u have classical meanings. Let $x_0 \in \Omega$.

As u is a viscosity solution, it is then both a subsolution and a supersolution. In both cases, as u is regular, it can then be taken as a test function at x_0 for itself. We then have:

$$F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \geq 0$$

and

$$F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \leq 0$$

which gives the result for all $x_0 \in \Omega$.

□

There exists an equivalent characterization of being a viscosity solution without using test functions for solutions that possess second order subdifferentials or superdifferentials (see [2] for details). It is particularly useful when we want to test whether a given function is a viscosity solution or not. For example, it helps to show that the following function is a viscosity solution of (1) and, in fact, is the unique one :

$$u : \begin{cases} [0,1] \longrightarrow \mathbb{R} \\ x \in [0, \frac{1}{2}] \mapsto x \\ x \in]\frac{1}{2}, 1] \mapsto 1 - x \end{cases} \tag{5}$$

As a final comment, it is important to notice that by construction, viscosity solutions theory will not select the same solutions to solve $F = 0$ and $-F = 0$.

2.2 Passages to the limit and historical example of vanishing viscosity method.

We can wonder why the word "viscosity" is used in the previous definition. The historical reason is that, for first order Hamilton-Jacobi equations ($F(x, u, Du) = 0$), viscosity solutions have first been obtained by the vanishing viscosity method. The first purpose of this subsection is to illustrate in what way this method has historically inspired Definition 2.1.

Here is the idea. To study solutions of $F(x, u, Du) = 0$, let us consider the following equation in Ω for $\varepsilon > 0$:

$$-\varepsilon \Delta u_\varepsilon(x) + F(x, u_\varepsilon(x), Du_\varepsilon(x)) = 0. \tag{6}$$

We admit that $\forall \varepsilon > 0, \exists!\ u_\varepsilon \in C^2(\Omega, \mathbb{R}) \cap W^{1,\infty}(\Omega, \mathbb{R})$ solution of (6). We also admit that the Lipschitz constant does not depend on ε. Then, by Ascoli Theorem, up to a subsequence, $(u_\varepsilon)_{\varepsilon>0}$ uniformly converges to $u_0 \in C(\Omega, \mathbb{R})$. As it does not involve convergence for the derivatives, it does not enable to pass to the limit $\varepsilon \to 0$ in (6).

It would be very convenient to find a characterization of being a classical solution of (6) that does not depend on the derivatives of u_ε so as to pass to the limit. That is the object of the following theorem:

Theorem 2.4 (Characterization of being solution of (6))**.**
Let $v \in C^2(\Omega, \mathbb{R})$. Then, v is a classical solution of (6) if and only if for all $x_0 \in \Omega$ and all test function $\varphi \in C^2(\Omega, \mathbb{R})$ such that $v - \varphi$ attains a local minimum (resp. a local maximum) at x_0, we have:

$$-\varepsilon \Delta \varphi(x_0) + F(x_0, v(x_0), D\varphi(x_0)) \geq 0 \quad (\textit{resp.} \leq).$$

Remark 2.5 (A quick digression to the future)**.** *If we quickly have a look to the general Definition (2.1) for first and second order equations which historically came later, this theorem tells us that v is the classical solution u_ε of (6) if and only if v is the viscosity solution of the same equation. The above characterization is, in fact, retrospectively a particular application of Corollary 2.3. That suggests that viscosity solutions theory generalizes the historical approach.*

The proof is quite straightforward and uses the fact that the Laplacian is the trace of the Hessian matrix.

With such a formulation, it is then possible to pass "formally" to the limit $\varepsilon \to 0$ so as to introduce and understand the historic definition of viscosity solution of $F(x, u, Du) = 0$ (first order) for the uniform limit u_0 of $(u_\varepsilon)_{\varepsilon>0}$. It is precisely the same formulation as in Definition 2.1 provided F does not depend on X (with a density argument, we can extend the definition for test functions that are only in $C^1(\Omega, \mathbb{R})$). However, this formal reasoning, used to define a relevant notion, is not a proof of convergence of viscosity solutions in the sense of Definition 2.1 at all, as it neglects the differences between test functions for u_0 and for the sequence $(u_\varepsilon)_{\varepsilon>0}$.

Nevertheless, we get the following theorem regarding the general Definition 2.1:

Theorem 2.6 (Passage to the limit in (6))**.** *Each accumulation point of the sequence $(u_\varepsilon)_{\varepsilon>0}$ is a viscosity solution of $F(x, u, Du) = 0$. If this equation has a unique viscosity solution, then there is only one accumulation point to the sequence and all the sequence $(u_\varepsilon)_{\varepsilon>0}$ converges to this value.*

Proof. Let u_0 be an accumulation point of the sequence $(u_\varepsilon)_{\varepsilon>0}$ (without changing its name when considering a corresponding converging subsequence). We will only perform the proof for the subsolution part, the supersolution part being exactly the same. Let us consider $x_0 \in \Omega$. By definition of Ω, there exists $r > 0$ such that $B(x_0, r) \subset \Omega$ (B denoting a closed ball).

Let us consider $\varphi \in C^2(\Omega, \mathbb{R})$ such that $u_0 - \varphi$ has a strict local maximum (to ensure uniqueness) at x_0 (see remark 2.2). Upon restricting the closed ball (that means eventually choosing a smaller r), we can suppose that the strict extremum is valid on the whole closed ball.

Given $\varepsilon > 0$, $u_\varepsilon - \varphi$ is continuous on the compact set $B(x_0, r)$ and therefore attains a maximum (not necessarily unique). Let us denote it by C_ε and p_ε one corresponding preimage, that means:

$$C_\varepsilon = \max_{B(x_0,r)} u_\varepsilon - \varphi = (u_\varepsilon - \varphi)(p_\varepsilon).$$

We define $\varphi_\varepsilon := \varphi + C_\varepsilon$. By definition of C_ε, we have:

$$u_\varepsilon - \varphi_\varepsilon \leq 0$$

on $B(x_0, r)$ with equality at p_ε.

Then, by applying Theorem 2.4 on u_ε with φ_ε as test function, and as C_ε is a constant, we get:

$$-\varepsilon \Delta \varphi(p_\varepsilon) + F(p_\varepsilon, u_\varepsilon(p_\varepsilon), D\varphi(p_\varepsilon)) \leq 0. \tag{7}$$

Whatever the choice for p_ε at fixed ε is, the whole sequence $(p_\varepsilon)_{\varepsilon>0}$ converges to x_0. By contradiction, if it is not the case, there exists ε_0 and a subsequence, (p'_ε) say, such that $|p'_\varepsilon - x_0| > \varepsilon_0$. By compactness of $B(x_0, r)$, we can extract another subsequence still called (p'_ε) that converges to $x_1 \in B(x_0, r)$. In particular, $|x_1 - x_0| \geq \varepsilon_0 > 0$.

By definition of p'_ε:

$$(u'_\varepsilon - \varphi)(p'_\varepsilon) \geq (u'_\varepsilon - \varphi)(y) \qquad \forall y \in B(x_0, r).$$

We write $u'_\varepsilon(p'_\varepsilon) = [u'_\varepsilon(p'_\varepsilon) - u_0(p'_\varepsilon)] + [u_0(p'_\varepsilon) - u_0(x_1)] + u_0(x_1)$ and use the uniform convergence and the continuity of u_0 and φ to pass to the limit in the previous inequality. We end up with:

$$(u_0 - \varphi)(x_1) \geq (u_0 - \varphi)(y) \qquad \forall y \in B(x_0, r)$$

and hence by uniqueness $x_1 = x_0$ which is impossible.

We also have: $\lim_{\varepsilon \to 0} C_\varepsilon = (u_0 - \varphi)(x_0)$ by definition of p_ε.

We can then easily pass to the limit $\varepsilon \to 0$ in (7) using the continuity of u_0, F, φ and its derivatives to get:

$$F(x_0, u(x_0), D\varphi(x_0)) \leq 0.$$

This ends the first part of the proof.

Let us now suppose that $F(x, u, Du) = 0$ has a unique viscosity solution. By contradiction, if the whole sequence does not converge to u_0, there exists ε_1 and a subsequence, (u'_ε) say, such that $|u'_\varepsilon - u_0|_\infty > \varepsilon_1$. By Ascoli Theorem and the first part of the proof, we can extract another subsequence, still called (u'_ε) that converges to the unique solution of $F(x, u, Du) = 0$, that is u_0. Passing to the limit in the previous inequality with that subsequence gives the contradiction. □

This theorem can be used to show a convergence result in the vanishing viscosity method used in (1) towards the function defined in (5).

This theorem will in fact be a corollary of the good compatibility of viscosity solutions theory with passages to the limit. This compatibility will be stated in Theorem 2.7 and is the second purpose of this subsection.

Theorem 2.7 (Passages to the limit)**.** *Let* $(F_\varepsilon)_{\varepsilon>0}$ *be a sequence of proper Hamiltonians and* $(u_\varepsilon)_{\varepsilon>0}$ *be a sequence of corresponding subsolutions (resp. supersolutions). If* $F_\varepsilon \to F$ *in* $L^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n))$ *and* $u_\varepsilon \to u$ *in* $L^\infty(\Omega)$*, then* u *is a subsolution (resp. supersolution) of* (3)*.*

Proof. The proof is very similar to the previous one. It is detailed in Theorem 3.1 of [5]. □

It is important to notice that this theorem enables us to pass to the limit in equations that are non-linear in the derivatives, with uniform convergence of functions (and not of their derivatives) as the only requirement. Another thing to remark is that the uniform convergences are separately considered in the assumptions.

Theorem 2.6 is a corollary of Theorem 2.7 when choosing $F_\varepsilon(x, r, p, X) = -\varepsilon Tr(X) + F(x, r, p)$.

We now look at classical features of PDEs, such as existence and uniqueness for solutions. We begin with existence. We will start by giving an alternative definition for viscosity solutions to cover more general cases.

2.3 Perron's method

Let u be a locally bounded function in Ω. We define its upper semicontinuous envelope u^* as:

$$u^*(x) = \limsup_{y \to x} \; u(y) \tag{8}$$

and similarly for its lower semicontinuous envelope u_* by replacing supremum limit by infimum limit. We can see that u is continuous if and only if $u^* \leq u_*$.

We now come to an alternative definition of viscosity solutions for discontinuous functions:

Definition 2.8 (Discontinuous viscosity solutions)**.** *We say that u is a discontinuous viscosity solution of* (3) *if u_* and u^* are respectively a supersolution and a subsolution of* (3)*.*

We remark that if u is continuous and is a discontinuous viscosity solution then it will be a viscosity solution in the sense of Definition 2.1.

We now state a lemma and Perron's method as a deriving theorem without giving their proofs that are given in [6].

Lemma 2.9 (Supremum of subsolutions)**.** *Let consider $(u_\alpha)_{\alpha \in A}$ a family of functions that is uniformly locally bounded from above. If $(u_\alpha)_{\alpha \in A}$ are subsolutions of* (3) *then $u := \left(\sup_{\alpha \in A} u_\alpha \right)^*$ is a subsolution of* (3)*.*

Now here is the main theorem dealing with existence of viscosity solutions:

Theorem 2.10 (Perron's method)**.** *Let u and v be respectively a subsolution and a supersolution of* (3) *such that $u \leq v$ in Ω. Then there exists a discontinuous viscosity solution w of* (3) *such that $u \leq w \leq v$.*

In Perron's method, we suppose pre-existence of subsolutions and supersolutions that are reffered to as barriers. Those are not very difficult to construct in general, though sometimes technical and details can be found in Example 4.6 of [3] or in Lemma 2.11 of [4].

We now end with comparison principles. Those are indeed linked with uniqueness of solution.

2.4 Comparison principles

Here we will give three different comparison principles. The two first ones come from [2] and the last one is inspired from [6]. The corresponding proofs are also detailed in those references.

The first one concerns first order Hamilton-Jacobi equations.

We consider the case where $F(x, r, p, X) = r + H(x, p)$ (that is to say $u + H(x, Du) = 0$) with H a Lipschitz continuous Hamiltonian with respect to the variable x such that:

$$|\partial_x H(x, p)| \leq C(1 + |p|) \qquad C > 0,\ \forall (x, p) \in \Omega \times \mathbb{R}^n. \tag{9}$$

Theorem 2.11 (A comparison principle for first order equations)**.**
Under the assumption (9), let u and v be respectively a subsolution and a supersolution of (3) in Ω such that $u \leq v$ on $\partial\Omega$. Then, $u \leq v$ in Ω.

Now, here is a comparison principle that concerns purely second order equations. We consider the case $F(x, r, p, X) = r + G(X) - f(x)$ (that is to say $u + G(D^2u) = f$) where f is a continuous function in Ω and G is elliptic degenerate.

Theorem 2.12 (A comparison principle for purely second order equations)**.**
Under the previous assumptions, let u and v be respectively a subsolution and a supersolution of (3) in Ω such that $u \leq v$ on $\partial\Omega$. Then, $u \leq v$ in Ω.

Here we suppose that F does not depend on the space variable x and that for some $\lambda > 0$, it verifies:

$$r > s \Rightarrow F(r, p, X) - F(s, p, X) > \lambda(r - s) \qquad \forall (r, s, p, X) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times S(n).$$

We then come to the following comparison principle:

Theorem 2.13 (A comparison principle for space independent second order equations)**.**
Under the previous assumptions, let u and v be respectively a subsolution and a supersolution of (3) in Ω such that $u \leq v$ on $\partial\Omega$. Then, $u \leq v$ in Ω.

Comparison principles actually deal with uniqueness as highlighted in the following corollary:

Corollary 2.14 (Uniqueness)**.** *Under a comparison principle and for a given Dirichlet boundary condition (u_1 say) there exists at most one viscosity solution of (3).*

Proof. Let u and v be two viscosity solutions of (3) such that $u = v = u_1$ on $\partial\Omega$. Then, u is a subsolution and v is a supersolution of (3) such that $u \leq v$ on $\partial\Omega$. By comparison principle, we have $u \leq v$ in Ω. Conversely, v is a subsolution and u a supersolution of (3) such that $v \leq u$ on $\partial\Omega$. This time we get $v \leq u$ in Ω which finally gives $u = v$ in Ω. □

Comparison principles are also useful in Perron's method as they enable to recover continuity of the constructed solutions. Indeed, upon having a comparison principle, for a given solution u such that $u^* \leq u_*$ on $\partial\Omega$ then we have $u^* \leq u_*$ in Ω and thus u is continuous.

We will perform a proof of a comparison principle in the upcoming section where we will study a non-local equation that is very close to Hamilton-Jacobi equations.

I have also started to study Hamilton-Jacobi equations with discontinuous Hamiltonians and to see related concepts such as half-relaxed limits that have been introduced by Guy Barles and Benoit Perthame. I will further study it in my PhD thesis and some features can be found in [2] or in [5].

3 Study of a particular model

3.1 Definitions and notations

We are given here a 1D model in space and time. For $T > 0$, we study viscosity solutions of the following problem:

$$\begin{cases} u_t = V(u(x+1,t) - u(x,t)) & \forall (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x) & \forall x \in \mathbb{R}. \end{cases} \tag{10}$$

We make the following assumptions on V and u_0.

$$\begin{cases} V \text{ is a globally Lipschitz continuous and nondecreasing function on } \mathbb{R}, \\ u_0 \text{ is a globally Lipschitz continuous function on } \mathbb{R}. \end{cases} \tag{11}$$

Let us denote $[u]$ the function: $(x,t) \mapsto u(x+1,t) - u(x,t)$ and we set $J = \mathbb{R} \times [0,T)$.

In the remaining part of the section, we will use the following definitions when we will refer to viscosity solutions and test functions for this non-local equation.

Definition 3.1 (Viscosity solutions of (10))**.** *Assume (11) and let $u : J \to \mathbb{R}$ be a locally bounded function.*

- *The function u is said to be a subsolution (resp. a supersolution) of (10) in $\mathbb{R} \times (0,T)$ if u is upper semi-continuous (resp. lower semi-continuous) and for all $(x,t) \in \mathbb{R} \times (0,T)$ and all test function $\varphi \in C^1(\mathbb{R} \times (0,T))$ such that $u - \varphi$ attains a local maximum (resp. a local minimum) at (x,t), then we have:*
$$\varphi_t \leq V([u]) \quad (\textit{resp.} \geq) \quad \textit{at } (x,t).$$
- *u is said to be a subsolution (resp. a supersolution) of (10) in J if it is a subsolution (resp. a supersolution) in $\mathbb{R} \times (0,T)$ and if moreover it satisfies for all $y \in \mathbb{R}$*
$$u(y,0) \leq u_0(y) \quad (\textit{resp.} \geq).$$
- *A function u is said to be a viscosity solution of (10) in J if it is both a subsolution and a supersolution in J.*

Remark 3.2 (Link with Hamilton-Jacobi equations)**.** *This model is almost a Hamilton-Jacobi equation, and it would be if we had replaced $[u]$ by u_x. This link will be highlighted in subsection 4.2.*

3.2 Comparison principle for bounded solutions

Let us now state and prove a comparison principle for (10) :

Theorem 3.3 (A comparison principle for bounded functions)**.**
Under the assumptions (11), let u and v be respectively a bounded subsolution and a bounded supersolution of (10) in J. Then, we have $u \leq v$ in J.

Notation : Unless otherwise stated, all the constants, like upper bounds, will be denoted by C.

Proof of Theorem 3.3

By contradiction, we suppose that

$$M := \sup_{(x,t)\in J} u(x,t) - v(x,t) > 0.$$

Step 1 : Definition of another supremum N.

Here, we use a dedoubling variable method (with penalization).
Let us consider $\alpha, \delta, \varepsilon, \eta$ all positive constants and define h, a real-valued function on J^2 :

$$h(x,t,y,s) = u(x,t) - v(y,s) - \frac{(t-s)^2}{2\delta} - \frac{(x-y)^2}{2\varepsilon} - \frac{\eta}{T-t} - \frac{\eta}{T-s} - \alpha\frac{x^2}{2} - \alpha\frac{y^2}{2}, \tag{12}$$

and N its supremum as:

$$N = \sup_{(x,t,y,s)\in J^2} h(x,t,y,s). \tag{13}$$

Here N is finite because $h \leq |u|_\infty + |v|_\infty$.
Afterwards, by comparison it is clear that for $(x,t,y,s) \in J^2$, we have:

$$\lim_{|(x,y,\frac{1}{T-t},\frac{1}{T-s})|\to+\infty} h(x,t,y,s) = -\infty. \tag{14}$$

Then, we can consider a maximizing sequence (x_n, t_n, y_n, s_n) that belongs to a compact set $K_{\alpha,\eta} \subset J^2$ and that converges to some $(x', t', y', s') \in K_{\alpha,\eta}$.
By definition, we have:

$$\lim_{n\to+\infty} h(x_n, t_n, y_n, s_n) = N$$

By using the upper semi-continuity of h for this sequence, we get:

$$h(x', t', y', s') = N. \tag{15}$$

Remark 3.4 (Parametric dependence)**.** *Even we did not write it explicitly, the reader has to bear in mind that (x', t', y', s') and N depend on all the parameters $\alpha, \delta, \varepsilon, \eta$, unlike M.*

Step 2 : Comparison between M and N

Let us now show that for small values of the parameters, we have:

$$N \geq \frac{M}{2} \tag{16}$$

M being a supremum, for all $r > 0$, there exists $(x_r, t_r) \in J$ such that $(u - v)(x_r, t_r) \geq M - r$.

Moreover, N is an upper bound of h on J^2 and so:

$$N \geq h(x_r, t_r, x_r, t_r) \geq M - r - \alpha x_r^2 - \frac{2\eta}{T - t_r}.$$

For fixed r, we choose η and α as follows:

$$\begin{cases} \eta \leq \frac{r(T-t_r)}{2} \\ \alpha \leq \frac{r}{x_r^2}. \end{cases} \tag{17}$$

Hence we get $N \geq M - 3r$ and by choosing $r = \frac{M}{6} > 0$, we get the desired result.

Step 3 : A-priori estimates on (x', t', y', s')

First, we remind that:

$$0 < \frac{M}{2} \leq N = h(x', t', y', s')$$

To obtain a-priori estimates on (x', t', y', s'), we bound N from above using the fact that u and v are bounded in (12).

We end up with all the following a-priori estimates:

$$\begin{cases} \frac{(t'-s')^2}{2\delta} \leq C \\ \frac{(x'-y')^2}{2\varepsilon} \leq C \\ \alpha(x'^2 + y'^2) \leq C \\ \frac{\eta}{T-t'} \leq C \\ \frac{\eta}{T-s'} \leq C. \end{cases} \tag{18}$$

Step 4 : Contradiction with initial conditions or viscosity inequalities

Case 1 : $t' = s' = 0$.

Here we use the initial conditions that are reminded here:

$$\begin{cases} u(x,0) \leq u_0(x) & \forall x \in \mathbb{R} \\ -v(x,0) \leq -u_0(x) & \forall x \in \mathbb{R}. \end{cases} \tag{19}$$

We then bound from above $N = h(x', t', y', s')$ by using the two previous inequalities respectively for x' and y' and by bounding from above some negative terms by zero:

$$N \leq u_0(x') - u_0(y') - \frac{(x'-y')^2}{2\varepsilon}.$$

.

Afterwards, we use the fact that u_0 is a Lipschitz continuous function (with $L > 0$ as Lipschitz constant) to get, by defining $p = x' - y'$:

$$0 < \frac{M}{2} \leq N \leq L|p| - \frac{p^2}{2\varepsilon}. \tag{20}$$

We hence deduce that:

$$0 < \frac{M}{2} \leq \max_{p \in \mathbb{R}_+} \left(Lp - \frac{p^2}{2\varepsilon} \right) = \frac{L^2 \varepsilon}{2}$$

which is impossible for ε small enough.

Case 2 : $s' > 0$ and $t' = 0$.

In this case, we need to write the dependence of (x', y', s') on δ explicitly, and we write it $(x'_\delta, y'_\delta, s'_\delta)$. Nevertheless, we recall that M does not depend on δ.

By similar considerations, we easily get:

$$\frac{M}{2} \leq u_0(x'_\delta) - v(y'_\delta, s'_\delta) - \frac{(x'_\delta - y'_\delta)^2}{2\varepsilon}. \tag{21}$$

By (18), for fixed α, x'_δ and y'_δ are in a compact set, and therefore, we can extract a subsequence (without changing its name) such that $(x'_\delta, y'_\delta) \to (x'_0, y'_0) \in \mathbb{R}^2$ when $\delta \to 0$. We then pass to the supremum limit when $\delta \to 0$ in (21), using the continuity of u_0 to get:

$$\frac{M}{2} \leq u_0(x'_0) - \frac{(x'_0 - y'_0)^2}{2\varepsilon} + \limsup_{\delta \to 0} \Big(- v(y'_\delta, s'_\delta) \Big).$$

Let us remind that $s'_\delta \to 0$ when $\delta \to 0$ by (18) and that v is lower semi-continuous to get

$$\liminf_{\delta \to 0} v(y'_\delta, s'_\delta) \geq v(y'_0, 0).$$

Moreover, combining it with the initial condition for v leads to:

$$\frac{M}{2} \leq u_0(x'_0) - u_0(y'_0) - \frac{(x'_0 - y'_0)^2}{2\varepsilon}.$$

And we exactly proceed as in (20) to find a contradiction.

Case 3 : $t' > 0$ and $s' = 0$. This case is totally analogous and we also end to a contradiction.

Last case : $t' > 0$ and $s' > 0$.
Here we will use viscosity inequalities.

By definition of N, we especially have :

$$N \geq h(x, t, y', s') \qquad \forall (x, t) \in J.$$

Hence, we can define a test function in J for the subsolution u, φ say, using the above inequality and (12):

$$\varphi : \begin{cases} J \longrightarrow \mathbb{R} \\ (x, t) \mapsto N + v(y', s') + \frac{(t-s')^2}{2\delta} + \frac{(x-y')^2}{2\varepsilon} + \frac{\eta}{T-t} + \frac{\eta}{T-s'} + \alpha \frac{x^2 + y'^2}{2}. \end{cases} \tag{22}$$

We have fixed y and s to y' and s' so as φ to be a smooth function of the variables $(x, t) \in J$ lying above u in J with equality at (x', t').

Similarly, we can define a test function for v, ϕ say, of the variables $(y, s) \in J$ (ϕ is this time below v in J with equality at (y', s')). Here, (x, t) is fixed to (x', t'):

$$\phi : \begin{cases} J \longrightarrow \mathbb{R} \\ (y, s) \mapsto u(x', t') - N - \frac{(t'-s)^2}{2\delta} - \frac{(x'-y)^2}{2\varepsilon} - \frac{\eta}{T-t'} - \frac{\eta}{T-s} - \alpha \frac{x'^2 + y^2}{2}. \end{cases} \tag{23}$$

Afterwards, we write the corresponding viscosity inequalities (see Definition 3.1).

For u we get:

$$\frac{t'-s'}{\delta}+\frac{\eta}{(T-t')^2}\leq V(u(x'+1,t')-u(x',t')). \tag{24}$$

For v we get:

$$\frac{t'-s'}{\delta}-\frac{\eta}{(T-s')^2}\geq V(v(y'+1,s')-v(y',s')). \tag{25}$$

We then add (24) to the opposite of (25) taking into account that $T-t'<T$ and $T-s'<T$:

$$\frac{2\eta}{T^2}\leq V(u(x'+1,t')-u(x',t'))-V(q) \tag{26}$$

where $q:=v(y'+1,s')-v(y',s')$. By definition of (x',t',y',s') we have:

$$h(x',t',y',s')\geq h(x'+1,t',y'+1,s')$$

and with (12) we end to:

$$u(x'+1,t')-u(x',t')\leq q+\alpha(x'+y'+1). \tag{27}$$

Let us remark that from (18), we have $\alpha^2x'^2\leq 2C\alpha$ and similarly for y' and then, $\alpha(x'+y'+1)=O(\sqrt{\alpha})$. Hence, we get:

$$u(x'+1,t')-u(x',t')\leq q+O(\sqrt{\alpha}). \tag{28}$$

Taking the image of (28) by V (nondecreasing function), substracting $V(q)$ and combining with (26) leads to:

$$\frac{2\eta}{T^2}\leq V(q+O(\sqrt{\alpha}))-V(q). \tag{29}$$

As V is a Lipschitz continuous function, we get:

$$\frac{2\eta}{T^2}\leq O(\sqrt{\alpha}) \tag{30}$$

and when $\alpha\to 0$, we obtain that $\eta\leq 0$ which is again contradictory. (This also works if V is only uniformly continuous).

Conclusion : The first assumption, that is $M>0$, is then false and this proves the comparison principle. □

Remark 3.5 (About the boundedness assumption)**.** *In the previous theorem, we used the fact that u and v are bounded for only three occurences. The first one to show that N is finite, the second and last ones to show (14) and (18). The rest of the proof does not depend on this assumption.*

The previous comment suggests that there may exist a generalization of the comparison principle for unbounded solutions with appropriate assumptions. This is the purpose of the next theorem.

3.3 Generalization for unbounded solutions

Let us replace the boundedness assumption on u and v by the following one (which is typically verified by solutions constructed by Perron's method):

$$\begin{cases} u(x,t) \leq u_0(x) + C_0 t & \forall (x,t) \in J \\ v(y,s) \geq u_0(y) - C_0 s & \forall (y,s) \in J \end{cases} \tag{31}$$

where $C_0 > 0$.

Then, we can extend the comparison principle for (10):

Theorem 3.6 (An extended comparison principle). *Under the assumptions (11) and (31), let u and v be respectively a subsolution and a supersolution of (10) in J. Then, we have $u \leq v$ in J.*

Before giving the proof of the theorem, we state a useful lemma that we will show afterwards.

Lemma 3.7 (Technical lemma). *Under the assumptions (11) and (31), there exists $C > 0$ such that:*

$$u(x,t) - v(y,s) \leq C\sqrt{1 + |x-y|^2} \qquad \forall (x,t,y,s) \in J^2 \tag{32}$$

Proof of Theorem 3.6

As already noticed before, there are only three changes in the proof. As the space penalizing terms are quadratic, and as u and v are of Lipschitz order in space like u_0, it is straightforward to check that N stays finite and that (14) still holds.

The last point consists in showing that the a-priori estimates in (18) are still true. Instead of bounding u and $-v$ from above separately, we use Lemma 3.7. As the function $(x,y) \mapsto C\sqrt{1+|x-y|^2} - \frac{|x-y|^2}{2\varepsilon}$ is bounded from above in $\mathbb{R}^2$, it is straightforward to obtain the a-priori estimates. However, this method makes the second estimate of (18) do not hold anymore but this one was never used in the proof of Theorem 3.3. □

Remark 3.8 (Direct proof for an article). *A direct proof of this theorem is given in Appendix A and might be part of a future article.*

Proof of Lemma 3.7

Let us consider θ, μ, C_1, C_2 all positive constants and define H, a real-valued function on J^2 :

$$H(x,t,y,s) = u(x,t) - v(y,s) - C_1 t - C_2\sqrt{1+|x-y|^2} - \frac{\mu}{T-t} - \frac{\mu}{T-s} - \theta\frac{x^2}{2} - \theta\frac{y^2}{2}, \tag{33}$$

and N' its supremum as:

$$N' = \sup_{(x,t,y,s) \in J^2} H(x,t,y,s). \tag{34}$$

We notice that if $N' \leq 0$, then Lemma 3.7 is true if we send $(\theta, \mu) \to (0,0)$ in this inequality.

By contradiction, we assume that $N' > 0$. As u and v are of Lipschitz order, by similar considerations as in (14), there exists a compact set $K_{\theta,\mu} \subset J^2$ and $(\bar{x}, \bar{t}, \bar{y}, \bar{s}) \in K_{\theta,\mu}$ such that:

$$N' = H(\bar{x}, \bar{t}, \bar{y}, \bar{s}). \tag{35}$$

Case 1 : $\bar{t} = 0, \bar{s} \geq 0$.

We use the initial condition for u and (31) for v:

$$0 < N' \leq u_0(\bar{x}) - u_0(\bar{y}) + C_0\bar{s} - C_2\sqrt{1 + |\bar{x} - \bar{y}|^2}. \tag{36}$$

As $\bar{s} \leq T$ and u_0 is Lipschitz continuous, we get:

$$0 < L|\bar{x} - \bar{y}| + C_0T - C_2\sqrt{1 + |\bar{x} - \bar{y}|^2} \tag{37}$$

which is false for C_2 large enough.

Case 2 : $\bar{s} = 0, \bar{t} \geq 0$.

This case is similar to the previous one.

Last case : $\bar{s} > 0, \bar{t} > 0$.

Here we use viscosity inequalities for u and v. By applying the same method as in (24) and (25) we obtain:

$$C_1 + \frac{\mu}{(T - \bar{t})^2} \leq V(u(\bar{x} + 1, \bar{t}) - u(\bar{x}, \bar{t})), \tag{38}$$

$$-\frac{\mu}{(T - \bar{s})^2} \geq V(v(\bar{y} + 1, \bar{s}) - v(\bar{y}, \bar{s})). \tag{39}$$

We then reproduce the computations from (26) to get the contradiction $C_1 \leq 0$ (or $\mu \leq 0$) by using again the uniform continuity of V and letting $\theta \to 0$ provided that $\theta(x' + y' + 1) = O(\sqrt{\theta})$. Let us show that estimate to conclude.

We use (31) to get:

$$0 < N' \leq u_0(\bar{x}) - u_0(\bar{y}) + C_0(\bar{t} + \bar{s}) - C_2\sqrt{1 + |\bar{x} - \bar{y}|^2} - -\theta\frac{\bar{x}^2}{2} - \theta\frac{\bar{y}^2}{2}. \tag{40}$$

As u_0 is Lipschitz continuous, and $\max(\bar{t}, \bar{s}) \leq T$, we get:

$$0 < L|\bar{x} - \bar{y}| + 2C_0T - C_2\sqrt{1 + |\bar{x} - \bar{y}|^2} - -\theta\frac{\bar{x}^2}{2} - \theta\frac{\bar{y}^2}{2}. \tag{41}$$

For C_2 large enough, we conclude that:

$$\theta(\bar{x}^2 + \bar{y}^2) \leq C \tag{42}$$

which is sufficient to prove the desired estimate.

Conclusion : The first assumption, that is $N' > 0$, is then false and this proves Lemma 3.7. □

Similarly, we could adapt Perron's method to this non-local equation and this will be done later in my PhD thesis. Now, let us describe the models that will be studied in my PhD thesis. We will especially see their link with the previous analysis.

4 Traffic flow modeling and introduction to PhD thesis

4.1 The microscopic model

For the microscopic model, which is our starting point, we consider an infinite 1D road with equally spaced lights. Therefore, to transpose mathematically their presence and their influence on drivers' velocities, we can choose a periodic function in time and space, ψ say, which take values in $[0;1]$. $\psi \equiv 1$ corresponds to the case where there is no light. ψ can never be strictly superior to 1 as lights are supposed to have a blocking effect on traffic (ψ being a weight function that moderates drivers' velocities).

To clarify the upcoming assumptions on ψ, let us first consider that the distance d between lights and their common time period are respectively rescaled to one meter and one second, so that ψ is 1-periodic in space and time. Each light can then be placed at a given $k \in \mathbb{Z}$ on the real line.

The effect of each light is mainly local, we will suppose that it is contained in an open interval around each light, $i_r(k) =]k-r; k+r[$, where $r > 0$ is chosen to be constant for all lights, and r has to be stricly inferior to d, $r \in \left]0; \frac{1}{4}\right[$ for example (otherwise for $r \geq d$, the lights are too close and interact with each other). Between two lights, there is then a non-empty free zone where ψ identically equals 1, that is where there is no limiting effect on the traffic. The whole free zone on the real line, that is the complementary of the union of all the $i_r(k)$ for $k \in \mathbb{Z}$, is denoted by L. Here is a picture to understand the situation better:

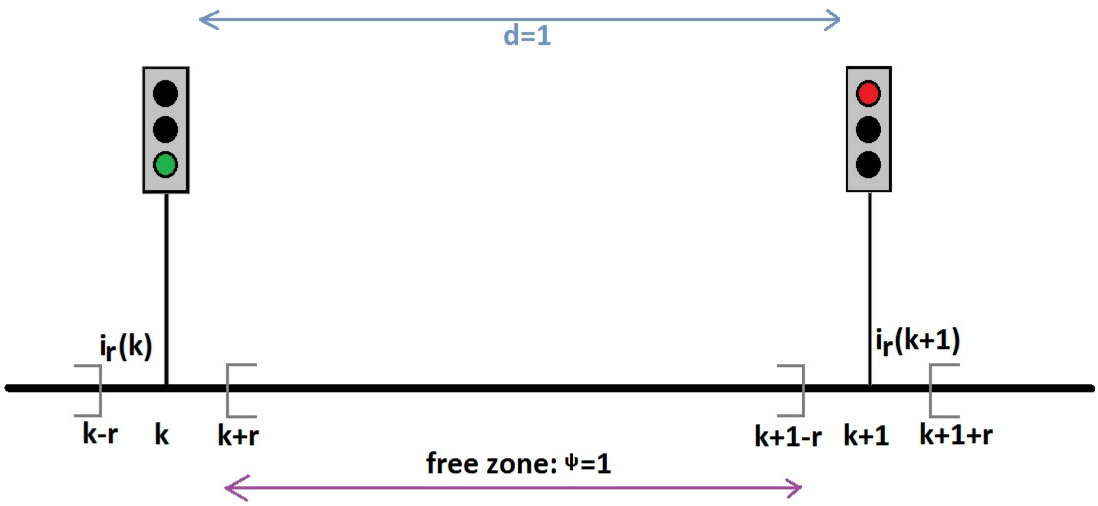

Figure 4: Sketch of the different zones.

We can remark that transposing the presence of a light on the road into a mathematical function is different from transposing the light itself into such a function, that is without considering the effect on its local environment. A fixed and isolated light can be seen as a function of the time variable, one-periodic in our example, and which varies between 0 and 1, that is, a periodic continuous function, p say (the case of a discontinuous piecewise constant function may be studied later in my PhD thesis as more technical). To be consistent, we then expect to have $\psi(k,t) = p(t)$ for all $k \in \mathbb{Z}$ and $t \in \mathbb{R}_+$, that means that a car precisely placed in front of a light, will be blocked if the light is red and will move forward normally if the light is green, as if there were no lights.

The local effect of the light is explained by the fact that if a driver sees a red light at a given $x = k$, then he will brake and will spend some time to retrieve his initial velocity after the light is becoming green. This is the case when the light is not "too far away" from him (the driver is in $i_r(k)$), otherwise he does not have any reason to stop, the latter case corresponding to the free zone. In each $i_r(k)$, ψ will take values in the whole interval $[0;1]$, so as to take into account the blocking effect of the light placed at $x = k$. To ensure a continuity between the two zones, it is reasonable to state that $\psi(k-r,t) = \psi(k+r,t) = 1$ for all $(k,t) \in \mathbb{Z} \times \mathbb{R}_+$, that is ψ is continuous in the space variable.

To sum up, we want that $\psi(k,t) = p(t)$, $\psi(k-r,t) = \psi(k+r,t) = 1$ and ψ takes its values in $[0;1]$ on $i_r(k)$. If we wanted ψ to be an affine function in space on $[k;k+r]$, we would simply take for $x \in [k;k+r]$,

$$\psi(x,t) = p(t) + (1-p(t))\frac{x-k}{r}.$$

However it does not respect the condition $\psi(k-r,t) = 1$. In order to verify it, we can simply take

$$\psi(x,t) = p(t) + (1-p(t))\frac{|x-k|}{r}$$

in each $i_r(k)$ which fulfills all the conditions. We can check that it is especially periodic in the space variable.

Finally, the simplest reasonable function ψ is the following one:

$$\begin{cases} \psi \geq 0 \\ \psi(x,t) = 1 \qquad \forall (x,t) \in L \times \mathbb{R}_+ \\ \psi(x,t) = p(t) + \frac{1-p(t)}{r}|x-k| \qquad \forall (k,x,t) \in \mathbb{Z} \times i_r(k) \times \mathbb{R}_+ \\ p(t+1) = p(t) \in [0,1]. \end{cases} \tag{43}$$

Now, let us try to give a simple traffic model on this road. Basically, the velocity of each driver is a nondecreasing and positive function, V say, of the distance that separates him from the preceding driver. It is reasonable to state V as bounded, like common car velocities. When there are lights on the road, this function V has to be weighted by ψ.

Let us suppose that the phase difference between two successive lights, is constant. This corresponds to c being constant in (44), where c can represent the propagation velocity of a green wave throughout lights when all the lights are initially red and one (for example at $k = 0$) suddenly becomes green. The phase difference between two consecutive lights is d/c, where $d = 1$. If the phase of the light at $k = 0$ is described by $\psi(0, t)$ then the phase of the k-th light will be given by $\psi(k, t - k/c)$. In first approximation, we choose a similar form for the phase for each driver between two lights, by replacing k by the driver's position in (44).

Given a sequence $(X_i)_{i\in\mathbb{Z}}$ of drivers' positions on the road, which are functions of time, the microscopic model then takes the form of an infinite coupled system of non-linear ODEs:

$$\frac{dX_i}{dt}(t) = \psi(X_i(t), t - X_i(t)/c)V(X_{i+1}(t) - X_i(t)) \qquad \forall i \in \mathbb{Z}. \tag{44}$$

We here see that this equation really looks like (10) when choosing $\psi \equiv 1$ (no lights).

If we take into account a common reaction time for all the drivers, τ say, we rather end to:

$$\frac{dX_i}{dt}(t+\tau) = \psi(X_i(t), t - X_i(t)/c)V(X_{i+1}(t) - X_i(t)) \qquad \forall i \in \mathbb{Z}. \tag{45}$$

This choice for ψ is not unique and I may consider others in my PhD thesis. Let us now try to have a look at homogenization.

4.2 Introduction to the homogenization process

The homogenization process corresponds to the rigorous passage from the Lagrangian microscopic scale (particle type) to the macroscopic Eulerian one (continuum type). Here is a schematic picture:

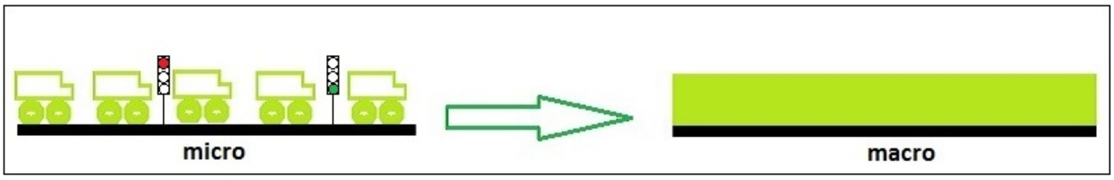

Figure 5: Schematic description of the homogenization process.

The key idea is first to embed the infinite microscopic ODE system into a single PDE (referred to as "pseudo-continuum"). We do this by interpolating the integer space index so that it is included in the space dependance of a space-time function. In order to go from the microscopic scale to the macroscopic one, we have to introduce a small parameter, ε say, so as to rescale properly the function and to take into account the microscopic oscillations with high frequency.

Hence, we define the function u^ε as follows (the space variable will be denoted as x):

$$u^\varepsilon(x,t) = \varepsilon X_{\lfloor \frac{x}{\varepsilon} \rfloor}(\frac{t}{\varepsilon}). \tag{46}$$

Substituting in (44) (or alternatively in (45)) leads to the following equation that is very close to (10):

$$\partial_t u^\varepsilon(x,t) = \psi\left(\frac{u^\varepsilon(x,t)}{\varepsilon}, t - \frac{u^\varepsilon(x,t)}{c\varepsilon}\right) V\left(\frac{u^\varepsilon(x+\varepsilon,t) - u^\varepsilon(x,t)}{\varepsilon}\right) \tag{47}$$

Therefore, we will be able to show the well-posedness of the initial microscopic model with the adaptations of viscosity solutions theory for non-local PDEs.

Let us introduce a slight change of functions. We define φ as: $\varphi(x,t) = \psi(x, t - \frac{x}{c})$. Then we get:

$$\partial_t u^\varepsilon(x,t) = \varphi\left(\frac{u^\varepsilon(x,t)}{\varepsilon}, t\right) V\left(\frac{u^\varepsilon(x+\varepsilon,t) - u^\varepsilon(x,t)}{\varepsilon}\right). \tag{48}$$

Finally, the homogenization process will consist in passing to the limit $\varepsilon \to 0$ in (48) in the viscosity sense, if possible (as in Theorem 2.7). More precisely, we will have to show that the unique viscosity solution u^ε of (48) converges locally uniformly towards the unique viscosity solution, u^0 say, of the following Hamilton-Jacobi equation:

$$\partial_t u^0 = V^0(u^0_x) \tag{49}$$

where V^0 will be a function related to V. This is again very close to (10) (see remark 3.2). As expected in classical traffic flow theory, this is deeply linked to conservation laws of cars' densities like in Lighthill-Whitham-Richards models, see [8] and [9]. Here, the density ρ say will be the spatial derivative of u^0 and its corresponding conservation law is the spatial derivative of (49):

$$\rho_t + (-V^0(\rho))_x = 0 \tag{50}$$

where $-V^0$ is the corresponding flux function.

The more the drivers are close one to each other, the more the density will increase. Roughly speaking, we then have locally:

$$\rho \sim \frac{1}{X_{i+1}(t) - X_i(t)}.$$

Some natural questions can arise, for example: is homogenization always possible?

Let us consider the pursuit model with reaction time without lights, that is (45) with $\psi \equiv 1$. If a driver suddently brakes and if the following one reacts lately, this latter will be obliged to brake suddenly. He will then force the next ones to do so: this will create a traffic jam that will vanishes later elsewhere. This phenomena is called "accordion effect". Therefore, a huge reaction time can create microscopic instabilities that propagate and make a continuum/macroscopic model impossible to perform.

The first aim of my PhD thesis will be to study the effects of this reaction time on global traffic flow. More precisely, I will try to show that for "small" reaction times, a homogenization process can be performed and that for "huge" reaction times, it is not possible anymore.

The second purpose of my PhD thesis will be to consider the model with lights and study their effect on global traffic flow, that is performing an homogenization with lights. I will try to perform explicit calculus of flux for simple cases (for simple ψ) and numerical simulations for more general cases. In a second time, I will try to optimize the phase difference between lights so as to fludify traffic flow, that is, to maximize flux in some sense. Basically, it will consist in enabling series of cars moving along several lights without being stopped, that is called "green waves". Microscopic models will be chosen between the lights if those are "close" one to each other and macroscopic models in the opposite case.

The lights, more precisely their phase difference which is more or less represented by the constant c, will also have some effects on homogenization. Another natural question is: Are there several kinds of homogenizations?

In the model with lights, we will be able to distinguish the periodic homogenization from the non-periodic one. Indeed, as ψ is one-periodic in space and time, we easily show that φ is space and time periodic if and only if $c \in \mathbb{Q}$. If it is not the case, then we go out from the periodic homogenization framework and I will also deal with that.

5 Conclusion

This report has tried to highlight the importance of viscosity solutions theory for the analysis of a huge class of equations, namely Hamilton-Jacobi equations. A way to apply and adapt this theory to traffic flow has been forseen in second and third parts and will be further developped in my PhD thesis.

Traffic flow has been studied in many different ways. That's why I may discover other fields and applications related to traffic during my PhD thesis, especially by discussing with researchers or PhD students from Ponts ParisTech school, Creteil University and IFSTTAR.

A lot of generalizations can be made from the studies I will perform and I might analyse some extensions in the future. For example, stochastic reaction times can be considered in order to better modelize randomness in drivers' behaviours. An interesting trail would be to extend our approach for roads with junctions.

This master's project has been a very good experience and I would like to thank again my two supervisors, Cyril Imbert and Régis Monneau for their continuous support, confidence and help, especially when I encountered difficulties in understanding some technical aspects of viscosity solutions theory. I hope that we will have a prolific collaboration during the three upcoming years.

In line with my previous master's projects, this one has enabled me to identify abilities which will be necessary in the research career I would like to pursue:

1. Patience when dealing with a new and unfamiliar concept.
2. Humility in front of problems, that is never underestimate difficulties before having studied them deeply.
3. Dedication when trying to derive a technical calculus.
4. Concision, clarity and rigour when redacting the report, so that global ideas are quickly understood by a huge range of readers.

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A Appendix: Direct proof of Theorem (3.6)

Theorem A.1 (Comparison principle)**.**
Under the assumptions (11) and (31), let u and v be respectively a subsolution and a supersolution of (10) in J. Then, we have $u \leq v$ in J.

Notation : Unless otherwise stated, all the constants, like upper bounds, will be denoted by C.

Before giving the proof of the theorem, we state a useful lemma that we will show afterwards.

Lemma A.2 (Technical lemma)**.** *Under the assumptions (11) and (31), there exists $C > 0$ such that:*

$$u(x,t) - v(y,s) \leq C\sqrt{1 + |x-y|^2} \qquad \forall (x,t,y,s) \in J^2 \tag{51}$$

Proof of Theorem A.1

By contradiction, we suppose that

$$M := \sup_{(x,t)\in J} u(x,t) - v(x,t) > 0$$

Step 1 : Definition of another supremum N.

Here, we use a dedoubling variable method (with penalization).
Let us consider $\alpha, \delta, \varepsilon, \eta$ all striclty positive constants and define h, a real-valued function on J^2 :

$$h(x,t,y,s) = u(x,t) - v(y,s) - \frac{(t-s)^2}{2\delta} - \frac{(x-y)^2}{2\varepsilon} - \frac{\eta}{T-t} - \frac{\eta}{T-s} - \alpha\frac{x^2}{2} - \alpha\frac{y^2}{2}, \tag{52}$$

and N its supremum as:

$$N = \sup_{(x,t,y,s)\in J^2} h(x,t,y,s) \tag{53}$$

As the space penalizing terms are quadratic, and as u and v are of Lipschitz order in space like u_0, it is straightforward to check that N is finite and that for $(x,t,y,s) \in J^2$, we have:

$$\lim_{|(x,y,\frac{1}{T-t},\frac{1}{T-s})|\to+\infty} h(x,t,y,s) = -\infty \tag{54}$$

Then, we can consider a maximizing sequence (x_n, t_n, y_n, s_n) that belongs to a compact set $K_{\alpha,\eta} \subset J^2$ and that converges to some $(x', t', y', s') \in K_{\alpha,\eta}$.
By definition, we have:

$$\lim_{n\to+\infty} h(x_n, t_n, y_n, s_n) = N$$

By using the upper semi-continuity of h for this sequence, we get:

$$h(x', t', y', s') = N. \tag{55}$$

Remark A.3 (Parametric dependence)**.** *Even we did not write it explicitly, the reader has to bear in mind that (x', t', y', s') and N depend on all the parameters $\alpha, \delta, \varepsilon, \eta$, unlike M.*

Step 2 : Comparison between M and N

Let us now show that:

$$N \geq \frac{M}{2} \tag{56}$$

M being a supremum, for all $r > 0$, there exists $(x_r, t_r) \in J$ such that $(u - v)(x_r, t_r) \geq M - r$. Otherwise, N is an upper bound of h on J^2 and so:

$$N \geq h(x_r, t_r, x_r, t_r) \geq M - r - \alpha x_r^2 - \frac{2\eta}{T - t_r}$$

For fixed r, we choose η and α as follows:

$$\begin{cases} \eta \leq \frac{r(T-t_r)}{2} \\ \alpha \leq \frac{r}{x_r^2} \end{cases} \tag{57}$$

Hence we get $N \geq M - 3r$ and by choosing $r = \frac{M}{6} > 0$, we get the desired result.

Step 3 : A-priori estimates on (x', t', y', s')

First, we remind that:

$$0 < \frac{M}{2} \leq N = h(x', t', y', s')$$

To obtain a-priori estimates on (x', t', y', s'), we bound N from above by using Lemma A.2. As the function $(x, y) \mapsto C\sqrt{1 + |x - y|^2} - \frac{|x-y|^2}{2\varepsilon}$ is bounded from above in $\mathbb{R}^2$, it is straightforward to obtain the following a-priori estimates:

$$\begin{cases} \frac{(t'-s')^2}{2\delta} \leq C \\ \alpha(x'^2 + y'^2) \leq C \\ \frac{\eta}{T-t'} \leq C \\ \frac{\eta}{T-s'} \leq C \end{cases} \tag{58}$$

Step 4 : Contradiction with initial conditions or viscosity inequalities

Case 1 : $t' = s' = 0$.

Here we use the initial conditions that are reminded here:

$$\begin{cases} u(x, 0) \leq u_0(x) & \forall x \in \mathbb{R} \\ -v(x, 0) \leq -u_0(x) & \forall x \in \mathbb{R} \end{cases} \tag{59}$$

We then bound from above $N = h(x', t', y', s')$ by using the two previous inequalities respectively for x' and y' and by bounding from above some negative terms by zero:

$$N \leq u_0(x') - u_0(y') - \frac{(x' - y')^2}{2\varepsilon}$$

.

Afterwards, we use the fact that u_0 is a Lipschitz continuous function (with $L > 0$ as Lipschitz constant) to get, by defining $p = x' - y'$:

$$0 < \frac{M}{2} \leq N \leq L|p| - \frac{p^2}{2\varepsilon} \tag{60}$$

We hence deduce that:

$$0 < \frac{M}{2} \leq \max_{p\in\mathbb{R}_+} \left(Lp - \frac{p^2}{2\varepsilon} \right) = \frac{L^2\varepsilon}{2}$$

which is impossible for ε small enough.

Case 2 : $s' > 0$ and $t' = 0$.

In this case, we need to write the dependence of (x', y', s') on δ explicitly, and we write it $(x'_\delta, y'_\delta, s'_\delta)$. Nevertheless, we recall that M does not depend on δ.

By similar considerations, we easily get:

$$\frac{M}{2} \leq u_0(x'_\delta) - v(y'_\delta, s'_\delta) - \frac{(x'_\delta - y'_\delta)^2}{2\varepsilon} \tag{61}$$

By (58), for fixed α, x'_δ and y'_δ are in a compact set, and therefore, we can extract a subsequence (without changing its name) such that $(x'_\delta, y'_\delta) \to (x'_0, y'_0) \in \mathbb{R}^2$ when $\delta \to 0$. We then pass to the supremum limit when $\delta \to 0$ in (61), using the continuity of u_0 to get:

$$\frac{M}{2} \leq u_0(x'_0) - \frac{(x'_0 - y'_0)^2}{2\varepsilon} + \limsup_{\delta\to 0} -v(y'_\delta, s'_\delta),$$

Let us remind that $s'_\delta \to 0$ when $\delta \to 0$ by (58) and that v is lower semi-continuous to get

$$\liminf_{\delta\to 0} v(y'_\delta, s'_\delta) \geq v(y'_0, 0)$$

Moreover, combining it with the initial condition for v leads to:

$$\frac{M}{2} \leq u_0(x'_0) - u_0(y'_0) - \frac{(x'_0 - y'_0)^2}{2\varepsilon}$$

And we exactly proceed as in (60) to find a contradiction.

Case 3 : $t' > 0$ and $s' = 0$. This case is totally analogous and we also end to a contradiction.

Last case : $t' > 0$ and $s' > 0$.
Here we will use viscosity inequalities.

By definition of N, we especially have :

$$N \geq h(x, t, y', s') \qquad \forall (x, t) \in J$$

Hence, we can define a test function in J for the subsolution u, φ say, using the above inequality and (12):

$$\varphi : \left\{ \begin{array}{l} J \longrightarrow \mathbb{R} \\ (x,t) \mapsto N + v(y', s') + \frac{(t-s')^2}{2\delta} + \frac{(x-y')^2}{2\varepsilon} + \frac{\eta}{T-t} + \frac{\eta}{T-s'} + \alpha\frac{x^2+y'^2}{2} \end{array} \right. \tag{62}$$

We have fixed y and s to y' and s' so as φ to be a smooth function of the variables $(x, t) \in J$ superior to u on J with equality at (x', t').

Similarly, we can define a test function for v, ϕ say, of the variables $(y, s) \in J$ (ϕ is this time inferior to v in J with equality at (y', s')). Here, (x, t) is fixed to (x', t'):

$$\phi : \left\{ \begin{array}{l} J \longrightarrow \mathbb{R} \\ (y,s) \mapsto u(x', t') - N - \frac{(t'-s)^2}{2\delta} - \frac{(x'-y)^2}{2\varepsilon} - \frac{\eta}{T-t'} - \frac{\eta}{T-s} - \alpha\frac{x'^2+y^2}{2} \end{array} \right. \tag{63}$$

Afterwards, we write the corresponding viscosity inequalities.

For u we get:

$$\frac{t'-s'}{\delta}+\frac{\eta}{(T-t')^2}\le V(u(x'+1,t')-u(x',t')) \tag{64}$$

For v we get:

$$\frac{t'-s'}{\delta}-\frac{\eta}{(T-s')^2}\ge V(v(y'+1,s')-v(y',s')) \tag{65}$$

We then add (64) to the opposite of (65) taking into account that $T-t'<T$ and $T-s'<T$:

$$\frac{2\eta}{T^2}\le V(u(x'+1,t')-u(x',t'))-V(q) \tag{66}$$

where $q:=v(y'+1,s')-v(y',s')$. By definition of (x',t',y',s') we have:

$$h(x',t',y',s')\ge h(x'+1,t',y'+1,s')$$

and with (52) we end to:

$$u(x'+1,t')-u(x',t')\le q+\alpha(x'+y'+1) \tag{67}$$

Let us remark that from (58), we have $\alpha^2x'^2\le 2C\alpha$ and similarly for y' and then, $\alpha(x'+y'+1)=O(\sqrt{\alpha})$. Hence, we get:

$$u(x'+1,t')-u(x',t')\le q+O(\sqrt{\alpha}) \tag{68}$$

Taking the image of (68) by V (nondecreasing function), substracting $V(q)$ and combining with (66) leads to:

$$\frac{2\eta}{T^2}\le V(q+O(\sqrt{\alpha}))-V(q) \tag{69}$$

As V is a Lipschitz continuous function, we get:

$$\frac{2\eta}{T^2}\le O(\sqrt{\alpha}) \tag{70}$$

and when $\alpha\to 0$, we obtain that $\eta\le 0$ which is again contradictory. (This also works if V is only uniformly continuous)

Conclusion : The first assumption, that is $M>0$, is then false and this proves the comparison principle. □

Proof of Lemma A.2

Let us consider θ,μ,C_1,C_2 all striclty positive constants and define H, a real-valued function on J^2 :

$$H(x,t,y,s)=u(x,t)-v(y,s)-C_1t-C_2\sqrt{1+|x-y|^2}-\frac{\mu}{T-t}-\frac{\mu}{T-s}-\theta\frac{x^2}{2}-\theta\frac{y^2}{2}, \tag{71}$$

and N' its supremum as:

$$N' = \sup_{(x,t,y,s)\in J^2} H(x,t,y,s) \tag{72}$$

We notice that if $N' \leq 0$, then Lemma A.2 is true if we send $(\theta,\mu) \to (0,0)$ in this inequality.

By contradiction, we assume that $N' > 0$. By similar considerations as in (54), it is clear that there exists a compact set $K_{\theta,\mu} \subset J^2$ and $(\bar{x},\bar{t},\bar{y},\bar{s}) \in K_{\theta,\mu}$ such that:

$$N' = H(\bar{x},\bar{t},\bar{y},\bar{s}) \tag{73}$$

Case 1 : $\bar{t} = 0, \bar{s} \geq 0$.

We use the initial condition for u and (31) for v:

$$0 < N' \leq u_0(\bar{x}) - u_0(\bar{y}) + C_0\bar{s} - C_2\sqrt{1+|\bar{x}-\bar{y}|^2} \tag{74}$$

As $\bar{s} \leq T$ and u_0 is Lipschitz continuous, we get:

$$0 < L|\bar{x}-\bar{y}| + C_0T - C_2\sqrt{1+|\bar{x}-\bar{y}|^2} \tag{75}$$

which is false for C_2 huge enough.

Case 2 : $\bar{s} = 0, \bar{t} \geq 0$.

This case is similar to the previous one.

Last case : $\bar{s} > 0, \bar{t} > 0$.

Here we use viscosity inequalities for u and v. By applying the same method as in (64) and (65) we obtain:

$$C_1 + \frac{\mu}{(T-\bar{t})^2} \leq V(u(\bar{x}+1,\bar{t}) - u(\bar{x},\bar{t})), \tag{76}$$

$$-\frac{\mu}{(T-\bar{s})^2} \geq V(v(\bar{y}+1,\bar{s}) - v(\bar{y},\bar{s})) \tag{77}$$

We then reproduce the calculus from (66) to get the contradiction $C_1 \leq 0$ (or $\mu \leq 0$) by using again the uniform continuity of V and letting $\theta \to 0$ provided that $\theta(x'+y'+1) = O(\sqrt{\theta})$. Let us show that estimate to conclude.

We use (31) to get:

$$0 < N' \leq u_0(\bar{x}) - u_0(\bar{y}) + C_0(\bar{t}+\bar{s}) - C_2\sqrt{1+|\bar{x}-\bar{y}|^2} - -\theta\frac{\bar{x}^2}{2} - \theta\frac{\bar{y}^2}{2} \tag{78}$$

As u_0 is Lipschitz continuous, and $max(\bar{t},\bar{s}) \leq T$, we get:

$$0 < L|\bar{x}-\bar{y}| + 2C_0T - C_2\sqrt{1+|\bar{x}-\bar{y}|^2} - -\theta\frac{\bar{x}^2}{2} - \theta\frac{\bar{y}^2}{2} \tag{79}$$

For C_2 huge enough, we conclude that:

$$\theta(\bar{x}^2+\bar{y}^2) \leq C \tag{80}$$

which is sufficient to prove the desired estimate.

Conclusion : The first assumption, that is $N' > 0$, is then false and this proves Lemma A.2. □